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Thermodynamics in 2+1 Dimensional QED with Chern-Simons Term

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Abstract

It is known that in the 2+1 dimensional quantum electrodynamics with Chern-Simons term, spontaneous magnetic field induces Lorentz symmetry breaking. In this paper, thermodynamical characters, especially the phase structure of this model are discussed. To see the behavior of the spontaneous magnetic field at finite temperature, the effective potential in the finite temperature system is calculated within the weak field approximation and in the fermion massless limit. We found that the spontaneous magnetic field never vanishes at any finite temperature. This result doesn't change even when the chemical potential is introduced. We also investigate the consistency condition and the probability that fermion stays in a lowest Landau level at finite temperature.

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1 Introduction

In a past decade, 2+1 dimensional gauge theory have been studied from a lot of motivation and many interesting and remarkable results have been found such as the chiral symmetry breaking [1], parity violation by radiatively inducing Chern-Simons term [2], and quantum Hall effects [3, 4]. Two years ago, it was found that in 2+1 dimensional QED with a bare Chern-Simons term, spontaneous magnetic field can be stable and induce spontaneous Lorentz symmetry breaking [5].

The effective potential, calculated by Hosotani [5] at fermion one loop level within the weak field approximation and in the fermion massless limit ($m_a \to 0$), is

$$V(B) = -\frac{e\kappa}{\pi^2}(\tan^{-1}\frac{4}{\pi})|B| + O(|B|^{3/2}), \tag{1}$$

where B is the magnetic field. The first term of RHS in (1) appears not in the tree level but in the one loop calculation. Of course generally, effective potential should be represented as a Lorentz invariant form and so the order parameter should not be B but $\mathrm{tr}F_{\mu\nu}F^{\mu\nu}$. Thus we can regard (1) as the effective potential only when we consider the model in the frame where electric field vanishes and only magnetic field exists. Looking at eq.(1), we can see immediately that true vaccum is not the point $B = 0$, but some lowest energy state with non-zero value of B. Then the spontaneous magnetic field $\langle B\rangle$ is induced and so Lorentz symmetry is spontaneously broken. And it can be confirmed that the Nambu-Goldstone boson in this symmetry breaking is transversely polarized gauge boson. Further, we can read in (1) that the constant κ which is the coefficient of Chern-Simons term makes the origin a singular point. Therefore it is a necessary condition for vanishing $\langle B\rangle$ that the first term of RHS in (1) vanishes.

In this paper we study the thermodynamical characters of the 2+1 dimensional QED with a bare Chern-Simons term, especially the effective potential in the finite temperature system [6, 7], to see the phase structure and to investigate the behavior of the spontaneous magnetic field in finite temperature with chemical potential. Our central concern is whether Lorentz symmetry is 'restored' in high temperature, namely whether the spontaneous magnetic field $\langle B\rangle$ vanishes like the case of the theory of the ferromagnetic material [8]. For this aim, we calculate the coefficient of $|B|$ in the effective potential at finite temperature. The calculation is carried out in the same approximation

scheme as in the ref.[5], namely, within the weak field approximation and in the fermion massless limit. As below, we show that the spontaneous magnetic field, which exists at $T = 0$, never vanishes at any finite temperature.

In the next section, to begin with, we start from the brief review of this model at zero temperature for later convenience. And then, after introducing temperature and chemical potential in the ordinary manner, we calculate the effective potential in the finite temperature system and discuss the behavior of the spontaneous magnetic field at finite temperature. We also discuss the consistency condition and the behavior of the filling factors at finite temperature.

2 2+1 Dimensional QED with Chern-Simons term

Consider a model described by the Lagrangian [5],

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \sum_a \bar{\psi}_a[\gamma_a^\mu(i\partial_\mu + q_a A_\mu) - m_a]\psi_a \ , \tag{2}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\nu \ , \tag{3}$$

where index a is the label of each fermion field and the metric and gamma matrices are defined

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1), \tag{4}$$

$$\gamma_a^\mu \equiv (\eta_a\sigma^3, i\sigma^1, i\sigma^2), \tag{5}$$

$$\eta_a \equiv \frac{i}{2}\mathrm{tr}\gamma_a^0\gamma_a^1\gamma_a^2 = \pm 1. \tag{6}$$

Note that there are two types of gamma matrices with $\eta_a = \pm 1$. We call fermions assigned $\eta_a = +1$ type gamma matrices η_+ fermion, and $\eta_a = -1$ type gamma matrices η_- fermion, respectively.

The solutions of Dirac equation for $\eta_+, q_a B > 0$ fermion are as follows,

$$\psi(x) = \sum_{n=0}^{\infty}\sum_{p=-\infty}^{\infty} a_{np}u_{np}(x) + \sum_{n=1}^{\infty}\sum_{p=-\infty}^{\infty} b_{np}^\dagger w_{np}(x) \tag{7}$$

$$E_n^a = \sqrt{m_a^2 + 2|q_a B|n} \tag{8}$$

where u_{np}, w_{np} are two component spinors defined in [5], and a_{np}, b_{np} are annihilation operators for fermion and anti-fermion, respectively. And we can get the η_- fermion solution by replacing $u_{np}(x) \leftrightarrow w_{np}(x)$ in (7). Note that there is an asymmetry of energy spectrum of fermion and anti-fermion at the lowest Landau level.

Hamiltonian operator and electric charge operator are

$$\begin{aligned} \hat{H}_a &= \sum_{n=0}^{\infty}\sum_p a_{np}^\dagger a_{np} E_n^a + \sum_{n=1}^{\infty}\sum_p b_{np}^\dagger b_{np} E_n^a - \sum_{n=1}^{\infty}\sum_p E_n^a, & (9) \\ \hat{Q}_a &= \left\{ \begin{array}{c} q_a \sum_p (a_{0p}^\dagger a_{0p} - \frac{1}{2}) \\ q_a \sum_p (-b_{0p}^\dagger b_{0p} + \frac{1}{2}) \end{array} \right\} \\ & \qquad + q_a \sum_{n=1}^{\infty}\sum_p (a_{0p}^\dagger a_{0p} - b_{0p}^\dagger b_{0p}) \,, \quad \left\{ \begin{array}{c} (\eta\epsilon(B) > 0) \\ (\eta\epsilon(B) < 0) \end{array} \right\}. & (10) \end{aligned}$$

If we put the same number of η_+ and η_- fermion ($N_f^+ = N_f^- = N_f$) in the model to impose chiral symmetry, and the filling factor as $\nu^+ = 1$, $\nu^- = 0$ respectively and set the all $q_a = e$, the vaccum expectation value of the total charge is

$$\begin{aligned} \langle 0 \,|\, \hat{Q} \,|\, 0 \rangle &= \sum_a \langle 0 \,|\, \hat{Q}_a \,|\, 0 \rangle \\ &= N_f^+ \langle 0 \,|\, \hat{Q}_+ \,|\, 0 \rangle + N_f^- \langle 0 \,|\, \hat{Q}_- \,|\, 0 \rangle = 2N_f \langle 0 \,|\, \hat{Q}_+ \,|\, 0 \rangle \\ &= 2N_f \langle 0 \,| \left\{ e \sum_p (a_{0p}^\dagger a_{0p} - \frac{1}{2}) + e \sum_{n=1}^{\infty} (a_{np}^\dagger a_{np} - b_{np}^\dagger b_{np}) \right\} |\, 0 \rangle \\ &= 2N_f e \sum_p \left(\nu^+ - \frac{1}{2} \right) = e\, N_p N_f \,, & (11) \end{aligned}$$

where $\nu^\pm$ are the filling factor which are defined in

$$\begin{aligned} \nu^+ &= \langle 0 \,| a_{0p}^\dagger a_{0p} |\, 0 \rangle & (12) \\ \nu^- &= \langle 0 \,| b_{0p}^\dagger b_{0p} |\, 0 \rangle \,, & (13) \end{aligned}$$

and N_p is the number of the degeneracy,

$$N_p = \frac{eB}{2\pi} \times (\text{area}) \,. \qquad (14)$$

Some distribution of fermion in the lowest Landau level which satisfied the consistency condition

$$\kappa = \frac{1}{2\pi}\sum_a q_a^2 \eta_a \left(\nu_a - \frac{1}{2}\right) , \tag{15}$$

can be stable in the presence of spontaneous magnetic field. In our choice, the condition (15) is satisfied and the spontaneous magnetic field can exist.

In this situation, the effective potential (1) is obtained within the weak field approximation and in the fermion massless limit.

3 Finite Temperature System

In high temperature, electron-positron pair creations may occur one after another and the total particle number may be increased, but the total electric charge Q in this system must be conserved. So we have to introduce the chemical potential μ_a in our calculation as a Lagrange multiplier of conserved Q_a [9]. And the effective potential is regarded as a thermodynamical potential V which is the function of B, T and μ_a.

$$J = -\frac{\partial V(B,T,\mu)}{\partial B} \tag{16}$$

$$Q_a = -\frac{\partial V(B,T,\mu)}{\partial \mu_a} \tag{17}$$

where J is an external field coupling with B, and Q_a is also an input parameter that we fix. Then, we can get

$$B = \alpha(T, J, Q_a) B_0, \tag{18}$$

where $\lim_{T\to 0}\alpha(T, J, Q_a) = 1$. Since the external J is not fixed, B remains independent variable. However if the electric charge Q_a is fixed by hand, μ_a is constrained by eq.(17) and then μ_a is a dependent variable and a function of B and T.

The B, T, Q_a dependence of μ_a can be determined from the equation

$$Q_a = \frac{1}{\beta}\frac{\partial \ln Z}{\partial \mu_a} . \tag{19}$$

where

$$
\begin{aligned}
Z &= \mathrm{tr} e^{-\beta(\hat{H}-\mu\hat{Q})} \\
&= \prod_a \left(e^{\frac{1}{2}\beta\mu_a q_a} + e^{-\frac{1}{2}\beta\mu_a q_q}\right)^{N_p} \\
&\quad \times \prod_{n=1}^{\infty}\left(1+e^{-\beta(E_n-\mu_a q_a)}\right)^{N_p} \prod_{n=1}^{\infty}\left(1+e^{-\beta(E_n+\mu_a q_a)}\right)^{N_p} , \qquad (20)
\end{aligned}
$$

where $\hat{H} = \sum \hat{H}_a$ and $\mu\hat{Q} = \sum \mu_a \hat{Q}_a$.

In our model, $q_a = e$, and $N_f^+ = N_f^- = N_f$ by chiral symmetry. And as setting $\nu^+ = 1, \nu^- = 0$ at zero temperature, $Q_+ = Q_- = eN_p^0/2$, where N_p^0 is N_p at zero temperature. Since the total electric charge in this system must be conserved even in finite temperature, identifing $Q(= N_f^+ Q_+ + N_f^- Q_-)$ at finite T with $\langle 0|Q|0\rangle$ at zero temperature, from (11), (19) and (20), we obtain

$$
\begin{aligned}
\frac{1}{2}B_0 &= B\left\{\frac{1}{2}\tanh\left(\frac{1}{2}e\beta\mu\right)\right. \\
&\quad \left. + \sum_{n=1}^{\infty}\frac{1}{e^{\beta(E_n-e\mu)}+1} - \sum_{n=1}^{\infty}\frac{1}{e^{\beta(E_n+e\mu)}+1}\right\} , \qquad (21)
\end{aligned}
$$

where B_0 is the magnetic field at $T = 0$. Temperature dependence of μ is shown in Figure 1. by solving this equation. Note that in our parameter choice, $\mu \equiv \mu^+ = \mu^-$.

We redefine the effective potential as

$$
\Delta V(B,\, \beta,\, \mu) = V(B,\, \beta,\, \mu) - V(B=0,\, \beta,\, \mu) . \qquad (22)
$$

Formally introducing T and μ according to Matsubara method [10], we obtain

$$
\begin{aligned}
\Delta V &= \frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2\vec{p}}{(2\pi)^2} \\
&\times \ln\left[\left(1+\Pi_0^{\mathrm{tot}}(p;B,\beta,\mu)\right)\right. \\
&\quad \times\left\{1+\frac{1}{\vec{p}^2+p_3^2}\left(p_3^2\Pi_0^{\mathrm{tot}}(p;B,\beta,\mu)+\vec{p}^2\Pi_2^{\mathrm{tot}}(p;B,\beta,\mu)\right)\right\} \\
&\qquad \left. +\frac{\left(\kappa-\Pi_1^{\mathrm{tot}}(p;B,\beta,\mu)\right)^2}{\vec{p}^2+p_3^2}\right] \\
&\quad -\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2\vec{p}}{(2\pi)^2}\ln\left\{B\to 0\right\} , \qquad (23)
\end{aligned}
$$

where $p_3 \equiv 2n\pi/\beta$, and $\Pi_0^{\rm tot}, \Pi_1^{\rm tot}, \Pi_2^{\rm tot}$ are three independent parts of the self energies in the one particle irreducible (1PI) gauge boson two point functions [5] and will be defined in the next section.

The coefficient of $|B|$ is obtained as the total derivative of ΔV by B at $B = 0$. To calculate this, it is convenient to divide this into two parts,

$$\left.\frac{d\Delta V}{dB}\right|_{B=0,\mu=\mu_0(T)} = \Delta V_a' + \Delta V_b' \,, \tag{24}$$

where

$$\Delta V_a' \;=\; \left.\frac{\partial \Delta V}{\partial B}\right|_{B=0,\mu=\mu_0(T)} \,, \tag{25}$$

$$\Delta V_b' \;=\; \left.\frac{\partial \Delta V}{\partial \mu}\,\frac{\partial \mu}{\partial B}\right|_{B=0,\mu=\mu_0(T)} \,. \tag{26}$$

In (24) etc., $\mu_0(T)$ denotes the value $\mu(B = 0, T)$ which is obtained from eq.(21). We can see that $\mu(B = 0, T) = 0$ at any temperature as below.

Since $d\Delta V/dB$ is continuous function of T, μ for $B \geq 0$,

$$\alpha(T, J = 0, Q) \neq 0 \;\; (T \leq T_c), \tag{27}$$

where $\alpha(T\,, J\,, Q)$ is defined in (18). In eq.(27), T_c is the critical temperature where the spontaneous magnetic field vanishes and that is just what we want to know. Thus when $T \leq T_c$, $B = 0$ means $B_0 = 0$. Then, from eq.(21), we get $\mu(B = 0, T) = 0$. As far as the coefficient of B calculated in this way is not zero, we can consider T_c as the higher value.

From now, we consider (24) $\sim$ (26) with $\mu_0(T) = 0$.

Then $\Delta V_a'$ is formally written as

$$\begin{aligned}\Delta V_a' \;=\; & -\frac{\kappa}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^2\vec{p}}{(2\pi)^2}\left.\frac{\partial\Pi_1^{\rm tot}(p;B,\beta)}{\partial B}\right|_{B=0} \\ & \times\Big[\Big\{1+\Pi_0^{\rm tot}(p;B=0,\beta)\Big\}\Big\{(p_3^2+\vec{p}^2)+ \\ & p_3^2\Pi_0^{\rm tot}(p;B=0,\beta)+\vec{p}^2\Pi_2^{\rm tot}(p;B=0,\beta)\Big\}+\kappa^2\Big]^{-1} .\end{aligned} \tag{28}$$

In eq.(28), we dropped the terms with $\Pi_1^{\rm tot}$ and with $\partial\Pi_{0,2}^{\rm tot}/\partial B$ because of zero within the weak field approximation, as calculated in the next section.

Thus we have to calculate only $\Pi_0^{\rm tot}, \Pi_1^{\rm tot}, \Pi_2^{\rm tot}$ at finite temperature to calculate $\Delta V_a'$.

4 Calculation of $\Pi_0^{\rm tot}, \Pi_1^{\rm tot}, \Pi_2^{\rm tot}$ at Finite Temperature

At zero temperature, Π_0, Π_1, Π_2 are defined from the 1PI gauge boson two point functions $\Gamma^{\mu\nu}$ [5],

$$
\begin{aligned}
\Gamma_a^{00} &= \vec{p}^{\,2}\Pi_0^a \\
\Gamma_a^{0j} &= p^0 p^j \Pi_0^a - i\epsilon^{jk} p_k \Pi_1^a \\
\Gamma_a^{i0} &= p^0 p^i \Pi_0^a + i\epsilon^{ik} p_k \Pi_1^a \\
\Gamma_a^{ij} &= \delta^{ij}(p^0)^2 \Pi_0^a + i\epsilon^{ij} p^0 \Pi_1^a - (\vec{p}^{\,2}\delta^{ij} - p^i p^j)\Pi_2^a
\end{aligned}
\tag{29}
$$

and

$$
\Pi_i^{\rm tot} = \sum_a \Pi_i^a = N_f \left(\Pi_i^{\nu=0} + \Pi_i^{\nu=1}\right) \quad (i = 0, 1, 2)\,. \tag{30}
$$

If we take the weak field approximation,

$$
\Gamma^{\mu\nu} = \Gamma^{\mu\nu(0)}(B) + O(B^2), \tag{31}
$$

$$
\begin{aligned}
\Gamma^{\mu\nu(0)}_{\nu=0}(p) &\equiv i\,q^2 \int \frac{d^3k}{(2\pi)^3} \mathrm{tr}\left[\gamma^\mu S_0^{(0)}(k)\gamma^\nu S_0^{(0)}(k-p)\right] && (32)\\
\Gamma^{\mu\nu(0)}_{\nu=1}(p,\,B) &= \Gamma^{\mu\nu(0)}_{\nu=0}(p) + \delta\Gamma^{\mu\nu(0)}(p,\,B), && (33)\\
\delta\Gamma^{\mu\nu(0)}(p,\,B) &\equiv i\,q^2 \int \frac{d^3k}{(2\pi)^3} \Big\{\mathrm{tr}\left[\gamma^\mu S_0^{(0)}(k)\gamma^\nu f(k-p)\right] + \\
&\qquad \mathrm{tr}\left[\gamma^\mu f(k)\gamma^\nu S_0^{(0)}(k-p)\right]\Big\} && (34)
\end{aligned}
$$

where

$$
S_0^{(0)}(k) \equiv \eta\,\frac{k^\mu\gamma_\mu}{k^2 - m^2 + i\epsilon}\,,\ \ f(k) \equiv 2\pi i\, e^{-\vec{k}^2 l^2}\delta(k_0 - m)(I + \sigma^3)\,, \tag{35}
$$

and $l^2 \equiv 1/|qB|$.

Now in the finite temperature system, since integral for time component changes to the modes summation, we don't have any ultraviolet singularities.

So we can take the weak field approximation here to calculate $\Gamma^{\mu\nu}_{\nu=0,1}$ up to B. Thus we get the finite temperature versions of $\Pi_0^{\rm tot}, \Pi_1^{\rm tot}, \Pi_2^{\rm tot}$.

$$
\begin{aligned}
\Pi_0^{\nu=0}(p\,;\,\beta,\,B) &= -\frac{1}{\beta}\frac{q^2}{\vec{p}^2}\sum_{n'}\int\frac{d^2\vec{k}}{(2\pi)^2} \\
&\times \frac{2\left[m^2+k_0(k_0-p_0)+\vec{k}^2-\vec{p}\cdot\vec{k}\right]}{\left(k_0^2-\vec{k}^2-m^2\right)\left[(k_0-p_0)^2-(\vec{k}-\vec{p})^2-m^2\right]}+O(B^2), \qquad (36)\\
\Pi_0^{\nu=1}(p\,;\,\beta,\,B) &= \Pi_0^{\nu=0}(p\,;\,\beta,\,B) \\
&-\frac{4\pi\, q^2}{\vec{p}^2}\frac{1}{\beta}\frac{1}{(2\pi)^2}\left\{(2m+p_0)\int d^2\vec{k}\frac{e^{-(\vec{k}-\vec{p})^2 l^2}}{p_0^2+2mp_0-\vec{k}^2+i\epsilon}\right. \\
&\left.\quad +\,(2m-p_0)\int d^2\vec{k}\frac{e^{-(\vec{k}+\vec{p})^2 l^2}}{p_0^2-2mp_0-\vec{k}^2+i\epsilon}\right\}, \qquad (37)\\
\Pi_1^{\nu=0}(p\,;\,\beta,\,B) &= +\frac{1}{\beta}\frac{2\,q^2}{\vec{p}^2}\sum_{n'}\int\frac{d^2\vec{k}}{(2\pi)^2} \\
&\times \frac{m\vec{p}^2-i(p_0-2k_0)(k_1p_2-k_2p_1)}{\left(k_0^2-\vec{k}^2-m^2\right)\left[(k_0-p_0)^2-(\vec{k}-\vec{p})^2-m^2\right]}+O(B^2), \qquad (38)\\
\Pi_1^{\nu=1}(p\,;\,\beta,\,B) &= \Pi_1^{\nu=0}(p\,;\,\beta,\,B) \\
&+\frac{4\pi\, i\, q^2}{\vec{p}^2}\frac{1}{\beta}\sum_{n'}\int\frac{d^2\vec{k}}{(2\pi)^2} \\
&\times\left[\frac{(ip_1-p_2)k_2-(p_1+ip_2)k_1}{k_0^2-\vec{k}^2-m^2+i\epsilon}\,\delta(k_0-p_0-m)e^{-(\vec{k}-\vec{p})^2l^2}\right. \\
&\left.+\frac{-\vec{p}^2+(p_1-ip_2)k_1+(ip_1+p_2)k_2}{(k_0-p_0)^2-(\vec{k}-\vec{p})^2-m^2+i\epsilon}\,\delta(k_0-m)e^{-\vec{k}^2l^2}\right], \qquad (39)\\
\Pi_2^{\nu=0}(p\,;\,\beta,\,B) &= -\frac{4q^2}{(\vec{p}^2)^2}\frac{1}{\beta}\sum_{n'}\int\frac{d^2\vec{k}}{(2\pi)^2} \\
&\times\frac{p_0^2\left[m^2+k_0(k_0-p_0)+\vec{k}^2-\vec{p}\cdot\vec{k}\right]+\vec{p}^2\left[m^2-k_0(k_0-p_0)\right]}{\left(k_0^2-\vec{k}^2-m^2\right)\left[(k_0-p_0)^2-(\vec{k}-\vec{p})^2-m^2\right]} \\
&+O(B^2), \qquad (40)\\
\Pi_2^{\nu=1}(\,p\,;\,\beta,\,B) &= \Pi_2^{\nu=0}(\,p\,;\,\beta,\,B\,), \qquad (41)
\end{aligned}
$$

where

$$k_0 \equiv ik_3 = i\,\frac{2n'+1}{\beta}\,\pi,\; p_0 \equiv ip_3 = i\,\frac{2n}{\beta}\,\pi,\; (n,\, n'\;:\;\text{integer}) \tag{42}$$

Some parts of fermion modes sum can be written in the complex integral forms by the useful of mathematical formulae [6].

After performing summation and complex integral, and taking the fermion massless limit, we finally obtain $\Pi^{\rm tot}$s at finite temperature,

$$\begin{aligned}
\Pi_0^{\rm tot}(p\,;\,B\,,\,\beta) &= \frac{\pi\kappa}{4}(\vec{p}^2+p_3^2)^{-1/2} + A_1(p\,,\,\beta) + O(B^2) && (43)\\
\Pi_1^{\rm tot}(p\,;\,B\,,\,\beta) &= 2\kappa e|B|\frac{1}{\vec{p}^2+p_3^2} + O(B^2) && (44)\\
\vec{p}^2\Pi_2^{\rm tot}(p\,;\,B\,,\,\beta) &= -p_3^2\Pi_0^{\rm tot}(p\,;\,B\,,\,\beta) \\
&\quad +(\vec{p}^2+p_3^2)^{1/2}\frac{\pi\kappa}{4} - A_2(p\,,\,\beta) + O(B^2)\,, && (45)
\end{aligned}$$

where A_1, A_2 are the parts of temperature effect from fermion loops.

$$A_1(p\,,\,\beta) = \frac{2\kappa}{\vec{p}^2}\int_0^\infty dk\frac{1}{e^{\beta k}+1} \times\left\{1-\left[\frac{\sqrt{(\vec{p}^2+p_3^2-4k^2)^2+16k^2p_3^2}+\vec{p}^2+p_3^2-4k^2}{2(\vec{p}^2+p_3^2)}\right]^{\frac{1}{2}}\right\} \tag{46}$$

$$\begin{aligned}
A_2(p\,,\,\beta) &= 2\kappa\frac{p_3^2}{\vec{p}^2}\int_0^\infty dk\frac{1}{e^{\beta k}+1} \\
&\quad +\kappa\,\frac{(\vec{p}^2+p_3^2)^{1/2}}{\vec{p}^2}\int_0^\infty dk\frac{1}{e^{\beta k}+1} \\
&\quad \times\frac{(*)}{\sqrt{(\vec{p}^2+p_3^2-4k^2)^2+16k^2p_3^2}} && (47)
\end{aligned}$$

where

$$\begin{aligned}
(*) \;\equiv\; \sqrt{2}\,\Big\{&(4k^2-p_3^2)\left[\sqrt{(\vec{p}^2+p_3^2-4k^2)^2+16k^2p_3^2}+\vec{p}^2+p_3^2-4k^2\right]^{1/2} \\
&-4kp_3\left[\sqrt{(\vec{p}^2+p_3^2-4k^2)^2+16k^2p_3^2}-(\vec{p}^2+p_3^2-4k^2)\right]^{1/2}\Big\}. && (48)
\end{aligned}$$

In (43) $\sim$ (45), if we take the limit $T=0$, $(\beta \to \infty)$, it can be easily confirmed that A_1, A_2 vanish and only the first term remain in each equation within weak field approximation and that our calculation reproduces the result in ref.[5].

5 Effective Potential with T and μ

To know the behavior of the spontaneous magnetic field in the finite temperature system, we have to investigate the effective potential and have to solve the gap equation generally. However to our aim seeing whether spontaneous magnetic field vanishes or not at finite temperature, we have only to know the behavior of the effective potential at neighborhood of $B=+0$, namely the behavior of the coefficient of $|B|$ as mentioned in previous section.

With the equations (43), (44), and (45) , we finally obtain after straightfoward calculation from eq.(28),

$$\Delta V_a' = -\frac{e\kappa}{2\pi^2}\alpha \sum_{n=-\infty}^{\infty} \int_0^\infty dx \frac{x^5}{F_n(x\,,\,\alpha)}, \tag{49}$$

where

$$\alpha \equiv \frac{2\pi T}{\kappa}\,, \quad x \equiv \frac{p}{\kappa}\,, \tag{50}$$

$$\begin{aligned} F_n(x\,,\,\alpha) \quad \equiv \quad & \left[x^2\left(x^2+(\alpha n)^2\right) + \frac{\pi}{4}x^2\sqrt{x^2+(\alpha n)^2} + \tilde{A}_1(x,\,\alpha,\,n)\right] \\ & \times \left[x^2\left(x^2+(\alpha n)^2\right) + \frac{\pi}{4}x^2\sqrt{x^2+(\alpha n)^2} - \tilde{A}_2(x,\,\alpha,\,n)\right] \\ & +x^4\left(x^2+(\alpha n)^2\right)\,, \end{aligned} \tag{51}$$

and,

$$\begin{aligned} & \tilde{A}_1(x\,,\,\alpha\,,\,n) \\ & \equiv \alpha\frac{\log 2}{\pi}\left(x^2+(\alpha n)^2\right) - \frac{\alpha}{\pi}\left(\frac{x^2+(\alpha n)^2}{2}\right)^{\frac{1}{2}} \int_0^\infty dy \frac{1}{e^y+1} \\ & \times \left\{\sqrt{\left(x^2+(\alpha n)^2 - \frac{\alpha^2}{\pi^2}y^2\right)^2 + \frac{4}{\pi^2}\alpha^4 n^2 y^2} + x^2+(\alpha n)^2 - \frac{\alpha^2}{\pi^2}y^2\right\}^{\frac{1}{2}}, \end{aligned} \tag{52}$$

$$
\begin{aligned}
&\tilde{A}_2(x\,,\,\alpha\,,\,n)\\
&\equiv \alpha^3 n^2 \frac{\log 2}{\pi} + \alpha^3 \left(\frac{x^2+(\alpha n)^2}{2}\right)^{\frac{1}{2}} \frac{1}{\pi}\int_0^\infty dy \frac{1}{e^y+1}\\
&\times\left\{\left(\frac{y^2}{\pi^2}-n^2\right)\left[\frac{\sqrt{\left(x^2+(\alpha n)^2-\frac{1}{\pi^2}\alpha^2 y^2\right)^2+\frac{4}{\pi^2}\alpha^4 n^2 y^2}+x^2+(\alpha n)^2-\frac{1}{\pi^2}y^2\alpha^2}{\left(x^2+(\alpha n)^2-\frac{1}{\pi^2}\alpha^2y^2\right)^2+\frac{4}{\pi^2}\alpha^4 y^2 n^2}\right]^{\frac{1}{2}}\right.\\
&\left.-\frac{2}{\pi}ny\left[\frac{\sqrt{\left(x^2+(\alpha n)^2-\frac{1}{\pi^2}\alpha^2 y^2\right)^2+\frac{4}{\pi^2}\alpha^4 n^2 y^2}-\left(x^2+(\alpha n)^2-\frac{1}{\pi^2}y^2\alpha^2\right)}{\left(x^2+(\alpha n)^2-\frac{1}{\pi^2}\alpha^2y^2\right)^2+\frac{4}{\pi^2}\alpha^4 y^2 n^2}\right]^{\frac{1}{2}}\right\}.
\end{aligned} \tag{53}
$$

In eq.(49), since we can analytically show in high temperature limit and in low temperature limit and numerically for all temperature that

$$F_n(x,\,\alpha) > 0,\ \forall\, x,\,\alpha,\,n, \tag{54}$$

we confirm that $\Delta V_a'$ is always negative.

$$\Delta V_a' < 0,\ \forall\ \alpha. \tag{55}$$

In high temperature region, to know the asymptotic behavior of $\Delta V_a'$, it is convenient to divide $\Delta V_a'$ into the zero mode part and the another part. Permuting integral variable x to αu in the zero mode part and to $\alpha n \sinh v$ in the another part, we can easily see that the each part have α^{-1} dependence asymptotically. Thus,

$$\Delta V_a' \propto -\frac{1}{T},\ \ (T\to\infty). \tag{56}$$

And we have analytically confirmed that in $T \to 0$, eq.(49) reproduces the result in ref.[5] using Euler - Maclaurin's mathematical formula [11],

$$\lim_{T\to 0}\Delta V_a' = -\frac{e\kappa}{\pi^2}\tan^{-1}\frac{4}{\pi}. \tag{57}$$

We also numerically calculated $\Delta V_a'$, (see Figure 2). We can see from Figure 2 that, except for near $T=0$, $\Delta V_a'(T)$ is monotonously increasing

to zero having the behavior $\sim -1/T$. This results seem to say that the spontaneous magnetic field would not vanish even in high temperature region, say $T_c = \infty$, if the second term, $\Delta V_b'$ wouldn't have any effect in $d\Delta V/dB$.

Of course, nextly $\Delta V_b'$ have also to be considered in our study. However, we can know the qualitative but essential behavior of $\Delta V_b'$ without direct calculation as below.

At first, from eq.(21), we can analytically check that in neighborhood of $B = 0$, $\partial\mu/\partial B$ is positive for all temperature and its B dependence is at most order minus one,

$$\frac{\partial\mu}{\partial B} \propto \frac{1}{B} + O(1) > 0, \quad (B \sim +0, \ \forall\, T). \tag{58}$$

And we can see

$$\frac{\partial\mu}{\partial B} \sim \begin{cases} \frac{1}{\sqrt{B}} & (T \to 0) \\ \frac{1}{T} & (T \to \infty) \end{cases} \tag{59}$$

from eq.(21). Further, since from eq.(17)

$$\begin{aligned} \frac{\partial\Delta V}{\partial\mu} &= -\frac{Q(B, T\,\mu) - Q(B=0, T, \mu)}{2N_f} \\ &= -\frac{e^2}{4\pi}\frac{B}{\alpha(T)} < 0, \end{aligned} \tag{60}$$

After all, $\Delta V_b'$ is negative for all T and

$$\Delta V_b' \to \ -0, \quad (T \to 0, \infty) \tag{61}$$

Therefore, $\Delta V_b'$ have essentially no any effects in the phase structure of the order parameter B, especially in the both limit $T \to 0, \ \infty$, $\Delta V_b'$ is to be zero (Figure 2).

6 Consistency Condition at Finite Temperature

In the case $T = 0$, as we are setting up the filling factor $\nu^+ = 1$, $\nu^- = 0$ (and $N_f^+ = N_f^- = N_f$ by chiral symmetry), the lowest Landau level degenerating to x^1 direction is filled by η_+ fermions and they keep the generating

spontaneous magnetic field B and there are no η_- fermions. Then the consistency condition (15) is of course satisfied,

$$\kappa = \frac{e^2}{2\pi} N_f. \tag{62}$$

This condition is the sufficient condition for having the spontaneous magnetic field B_0 stably as mentioned before.

In finite temperature system, the consistency condition (15) have to be rewitten as follows,

$$\kappa = \frac{e^2}{2\pi} \sum_a \eta_a \left(\langle \nu_a \rangle - \frac{1}{2} \right) + 2N_f e \langle \sum_{n=1}^{\infty} \sum_p \left(a_{np}^\dagger a_{np} - b_{np}^\dagger b_{np} \right) \rangle / B, \tag{63}$$

where

$$\langle x \rangle \equiv \frac{\mathrm{tr} x e^{-\beta(\hat{H} - \mu \hat{Q})}}{\mathrm{tr} e^{-\beta(\hat{H} - \mu \hat{Q})}}. \tag{64}$$

In our model, we have just seen that spontaneous magnetic field never vanishes and T_c has gone to infinity. In finite temperature, η_+ fermions which are in the ground states (the lowest Landau level) may be gradually excited to higher states. Then, naively $\langle \nu^+ \rangle$ seems to decrease as temperature grows. In fact, as the filling factor operators of $\eta_\pm$ fermion are defined at finite temperature,

$$\hat{\nu}^+ = \frac{1}{N_p} \sum_p a_{0p}^\dagger a_{0p} \, , \; \hat{\nu}^- = \frac{1}{N_p} \sum_p b_{0p}^\dagger b_{0p} \, , \tag{65}$$

we can easily calculate ensemble averages as follows,

$$\langle \nu^+ \rangle = \frac{1}{e^{-e\beta\mu} + 1} \, , \; \langle \nu^- \rangle = \frac{1}{e^{+e\beta\mu} + 1} \, . \tag{66}$$

Clearly, $\langle \nu^+ \rangle$ (, $\langle \nu^- \rangle$) is monotonously decreasing (, increasing) functions of T (see Figure 3.) and in high temperature limit, $\langle \nu^\pm \rangle \to 1/2$. This means, the probability that η_+ fermion exists in a ground state is multiplied by no any statistical weight in high temperature limit. The second term in eq.(63) comes from the charge which is shared by the excited fermion and anti-fermion and this can be got from eq.(21),

$$\frac{N_f e^2}{\pi} \left\{ \sum_{n=1}^{\infty} \frac{1}{e^{\beta(E_n - e\mu)} + 1} - \sum_{n=1}^{\infty} \frac{1}{e^{\beta(E_n + e\mu)} + 1} \right\}. \tag{67}$$

7 Conclusion

In this paper, we have examined the thermodynamical characters in 2+1 dimensional QED with Chern-Simons term, which the vaccum can be spontaneously magnetized by some preferable fermion distribution which satisfies the consistency condition. The behavior of the effective potential at finite temperature is calculated at one loop level within the weak field approximation and in the fermion massless limit. The coefficient of $|B|$ in the effective potential existing at zero temperature does not vanish even at finite temperature. Thus $B = 0$ point remains being singular and the lower vaccum at some $B \neq 0$ point exists.

Therefore we conclude that in our model the spontaneous magnetic field never vanishes at any finite temperature differently from the case of the theory of the four dimensional ferromagnetic material. We also discussed the consistency condition for having the spontaneous magnetic field at finite temperature and calculated the ensemble average of the filling factor - the probability that fermion stays in a ground state as the function of temperature and chemical potential.

Our results that the critical temperature T_c is infinite may seemingly look like strange if readers consider them in a realistic system, such as the 2+1 dimensional system in the 3+1 dimensional world. In this case, the hopping to 3+1 dimension must occur at some finite critical temperature and our considering situation must break down. However, as far as we stand on the 2+1 dimensional world, our results are consistent.

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FIGURE CAPTIONS

Fig.1: The behavior of the chemical potential μ as a function of B, T. The solid line is the curve when $eB/\kappa^2 = 1/2$. The dashed line is the curve when $eB/\kappa^2 = 1/4$.

Fig.2: T dependence of $\Delta V_a'$ and $\Delta V_a' + \Delta V_b'$. The solid line which represents $\Delta V_a'$ is monotonously increasing up to zero except for near $T = 0$. At the point A, $\Delta V_a'$ has the value of $-e\kappa/\pi^2 \tan^{-1} 4/\pi$ as in the eq.(57). The dashed line which approximately represents $\Delta V_a' + \Delta V_b'$ is always below the solid line and in the limit $T \rightarrow 0, \infty$, $\Delta V_b'$ turn to be zero rapidly.

Fig.3: The behavior of the filling factors for η_+, η_- fermion. The solid line represents $\langle \nu^+ \rangle$ and the dashed line represents $\langle \nu^- \rangle$.